Research Paper 1
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The Development of Abstract Mathematical Thinking in Primary School Children

Introduction

Many months ago, I submitted this proposal:

Our Montessori maths materials provide children with a wonderful foundation for mathematical thinking, but many Montessori teachers see and use the materials simply as teaching tools for specific skills. This topic interests me because I have seen Montessori classrooms where cards, worksheets or maths books are used regularly, supposedly because the children need daily assigned practice of specific mathematical operations or they will forget how to perform them. My own children briefly attended a (Montessori) school where they were required to complete "DOM" – Daily Oral Math – as part of their classroom activities.

This tendency to use the maths materials as didactic apparatus seems to lead to an overall approach to maths as a curriculum sequence, which becomes rigid and disregards appealing to the child's interest and the mathematical mind. Instead, children are ushered through the maths presentations as if they formed a linear curriculum.

I would like to research current literature on how mathematical abstraction develops in children, and then evaluate and discuss how this relates to the 6 – 12 maths materials, their presentation, and the information we give trainees in a 6 –12 course.

Without having done any preliminary reading, I naïvely thought the world of mainstream education and educational research would have definitively ascertained how mathematical abstraction occurs in children. I assumed that somewhere in the vast world of education, someone would have described and generalised the milestones along the way to abstract thinking in relation to any mathematical concept.

In our introduction to the maths album, when we discuss the topic of mathematical thinking and the materials we use, Montessori educators come from this perspective:

"Human Beings have been given an intelligence that allows us to understand the laws, orders and patterns of our universe. This intelligence also allows us to reason. We have minds capable of imagining things that aren't here in concrete form, and we're capable of making those things become real. So as we trace the history of human development, we see humans using the environment to invent and create things, to make things that simplify and improve our lives. This power of invention and creation also rests on law and order. As human beings invented and improved those objects, such as ways and tools for measurement, they were founding an exact science and using an inherent power of the human mind. We call this inherent power *The Mathematical Mind*."

I didn't imagine that our relatively small group would hold these big ideas uniquely – I expected the world of mathematicians and mathematics educators to have also examined this concept in some way.

Once I began reading, however, it became clear that there are only theories about abstraction, and those theories have evolved over time. There are research projects incorporating case studies focusing on a specific mathematical concept or concepts, such as learning about angles, ratios, percentages, or problem-solving using the four operations, and how children move toward abstraction within that particular sphere. But I found that, "Generally missing from the literature is research that examines the process by which students progress from one of these conceptual steps to a subsequent one," (Simon et al 2010)rather than the definitive picture I expected to find.

<u>What is Abstraction?</u>

The word *abstraction* has its roots in Latin, and means "drawing away" - *trahere*, drawing; and *ab/s*, away. It's usually associated with the characteristics of intelligence and higher-order thinking skills. It has also been defined as a process where the human being moves from a simple observation to the observation of a category, or a generalisation (Roth and Hwang 2006). In this process, a number of simple observations are classified in the mind, to form a category that now describes the group of observations. Abstraction as defined by Skemp's empirical model is, "an activity by which we become aware of similarities among our experiences." (Skemp 1986)

In the Montessori 6 - 12 training, we emphasise the difference between Great Stories and Key Lessons. The many possible Key Lessons are only necessary on an individual basis – the materials, demonstrations, stories and impressionistic charts are imaginative aids to help children explore and reach understanding of a concept through their own effort, be it research, manipulation, writing or creating.

In terms of the mathematics content of Cosmic Education, *abstraction* means the child is able to perform a mathematical process correctly, without using the Montessori materials. This same child should also be able to explain how something works, or what process we follow to find the answer, such as, "To find the area of a rectangle, we multiply the base times the height." Or, "When we multiply decimals, the number of decimal places in the product is equal to the sum of decimal places in the multiplier and the multiplicand." The child understands and can carry out this type of calculation using only paper and pencil and his or her mind. It's the Key Lessons that are related to the discussion of how a child reaches abstraction.

Mathematical research has investigated the nature and development of "advanced mathematical thinking," and Harel and Sowder (2005) suggested that rather than being seen as "thinking in advanced mathematics," this concept should instead be understood as a way of thinking mathematically that can begin as early as elementary school. This view takes into account ways of both thinking about and understanding mathematical ideas, and seems to come from the more global perspective Montessorians intend when discussing the Mathematical Mind. They argue that, "a student's growth in mathematical thinking is an evolving process …" and, "The term 'advanced' implies that a developmental process is involved. 'Advanced' is, therefore, not an absolute but a relative term, both in relation to a single way of thinking and in relation to different ways of thinking. The attainment of a certain way of thinking is not all or nothing, but gradual …" While the authors express this view specifically in relation to the goals of their study, it indicates a belief that humans have an inborn ability to view the world mathematically, and that this ability expands and develops over time.

Other authors have also discussed mathematical thinking as being intrinsic to human beings. Ginsburg and Amit (2008) summarise various authors in saying that young children, "deal with mathematics in everyday play, are curious about the subject, know something about it, and can learn interesting mathematics when they are taught … They deal spontaneously and sometimes joyfully with mathematical ideas."

Mathematics is an abstract tool to help define our world in terms of quantity, and it is based on grasping patterns that can be understood and later generalised. There is evidence that children "who recognise the structure of mathematical processes and representations acquire deep conceptual understanding." (Mulligan and Mitchelmore 2009) They examined children's awareness of pattern and structure in relation to early mathematical development. They asked, *Is there such a construct as awareness of mathematical pattern and structure* (AMPS) *that can be observed across a range of concept areas in early mathematics learning?* Their goal was to describe and measure AMPS, and, as a result, they defined 4 broad stages of structural development along a continuum:

- Pre-structural
- Emergent
- Partial
- Structural

In the final stage, the child's own representations correctly integrate numerical and spatial structural features. The authors concluded that there may be a general characteristic defined as AMPS displayed by young children across a wide range of early mathematical concepts, and that recognising this characteristic offers new ways to think about how to best support mathematical thinking.

Considering much of the initial work in the Montessori 6 - 12 maths albums – such as the Wooden Hierarchical Material, the Commutative and Distributive laws, work with multiples and factors or early squaring and cubing, and even the initial presentations of fractions and decimal fractions – supporting "awareness of pattern and structure" is inherent in the design and presentation. This would confirm the Montessori view that concrete manipulative work provides a critical foundation for abstract thinking – exactly what we mean by the statement that "mathematics needs to be presented to children in a way that they can carry out a sensorial exploration of the facts about mathematics," both in the Casa dei Bambini and the 6 - 12 classrooms.

So educational research recognises the existence of certain mathematical tendencies or capabilities in terms of mathematical learning – what we would call the Mathematical Mind. There is an awareness that understanding and supporting this capability could lead to increased mathematical achievement, and could help children deepen their understanding of mathematical ideas.

A Historical Approach to Thinking about Abstraction

Some of the important names that appear from the 1960s onward in the inquiry into abstraction are Kant, Vygotsky, Piaget, Freudenthal, Davidov, Walkerdine, and Steffe and Cobb. They are joined in this century by Hershkowitz, Schwartz and Dreyfus; Lobato; Mitchelmore and White; Harel and Sowder; and Simon.

I will elaborate on the some of their theories of abstraction and the terms used in the various models, before moving on to discuss conclusions related to mathematical abstraction when children are using concrete materials. After that, I'll examine the elements of some research-based teaching models. In each of these last 2 sections, I will relate the findings regarding mainstream mathematics education to what we do in a Montessori classroom. Then in the final section of this paper, I'll attempt to extrapolate ways we might apply these research-based conclusions to the information given during the maths lectures and demonstrations on the 6 - 12 training course.

Theories of Abstraction

The question of how abstraction occurs has interested scholars as far back as ancient times, but we begin more recently with Immanuel Kant, who said that we always abstract from something, which is an object of thought. (Kant 1968e) From this object of thought we create a new general concept that the conscious mind accepts. He used the term "reflective abstraction" because we reflect upon an initial abstract thought during the regrouping process. Kant stated that, "to turn appearances into concept, one has to compare, reflect, and abstract." (Kant 1968d)

Later, Vygotsky proposed a social interpretation of thinking and its origins. He said that objects always have a two-fold appearance – as their physical entity, and as an aspect of mental consciousness – and the thinker relates these 2 aspects to one another. (Vygotsky 1986) He believed learners move through 3 key social stages during their learning process. In relation to mathematics, a learner's everyday experiences in the "school" stage are built upon by "enlightened practice" (on the part of a teacher), which assists the learner to move toward abstraction starting from the mathematics of real life.

Piaget used Vygotsky's ideas as a foundation, and distinguished between simple (or empirical) and constructive (or reflective) abstraction. Examples of simple abstraction would be colour or weight, which are knowledge about a physical characteristic. Examples of constructive abstraction would be the concepts of "3," "more" or "less," which require the creation of a mental relationship. (Piaget 1970)

Over the course of 60 years' observation, he and others verified these ideas and concluded that children construct mathematical knowledge through constructive abstraction (Piaget 1971). Piaget proposed a Stage Theory of cognitive development. His theory states that cognitive development is a progressive reorganisation of the mental processes, resulting from both experience and maturation. The stages he described are the:

- Sensori-motor Stage
- Pre-operational Stage
- Concrete Operational Stage
- Formal Operation Stage

Piaget said that all learners pass through all the stages, and the later stages are built on the earlier ones. Interestingly, this sounds similar to Montessori's theory of the Four Planes of Development.

Freudenthal believed that the focal point for mathematical learning should be "mathematising" daily experiences by linking familiar concepts to mathematical ideas. In thinking about abstraction, he said that there's a difference between horizontal mathematising, in which the learner moves from the real world to mathematical symbols, and vertical mathematising, which encompasses the development of mathematical ideas. (Freudenthal 1968, 1973)

Davidov defined abstraction as "separating a quality common to a number of objects or situations from other qualities." He developed the idea of theoretical abstraction, in which the learner creates concepts that fit into a particular mathematical theory; in terms of abstraction, mathematics is treated as a specific system or category. (Davidov 1972/1990)

Walkerdine agreed with Vygotsky by saying that, "mathematical (thinking) cannot be dissociated from other social and material practices." (Walkerdine 1988). He questioned Piaget's theory that children as (mathematical) learners form their own development as they adapt their mental structures in relation to activity on the physical world.

At the start of the century, Steffe and Thompson (2000) began from the assumption that teaching does not cause learning, but that learning is a process that occurs within the individual. They were considered "among the leaders in reasearch that focuses on describing and specifiying the mathematical schemes and operations of students as conceptual learning progresses."

Hershkowitz, Schwarz and Dreyfus (2001) proposed the Nested RBC model of abstraction. They described 3 "epistemic actions" in the abstraction process: construction (C), recognition (R), and building-with (B). They consider construction to be the key to abstraction, and described it in this way: "when a novel structure 'enters the mind', it has to be cognised, or pieced together from components, usually simpler structures." This theoretical model also assumes that learners construct new abstractions by reorganising abstract concepts already possessed. This is viewed as a "vertical" model of abstraction, because a concept proceeds to a higher level of understanding in the learner's mind. The Nested RBC Model has received a significant amount of consideration.

Lobato and colleagues (2002, 2003) explained cognitive development as a transfer point, from concrete to abstract thinking. This transfer point is the constructive process for the learner, not simply a process of recognition.

Mitchelmore and White (2004) said that "the first phase of the abstraction process is the recognition of common features in a variety of different situations. In everyday experience these features may be superficial (e.g., colour), but in mathematics they are always structural (e.g., number). In the second phase, the similiarity that has been recognised becomes abstracted and forms a concept which in a sense embodies that similarity." Their Teaching for Abstraction model is based on this and recognises abstraction as a constructive process.

In a 2005 study, Harel and Sowder wrote about the DNR process for viewing mathematical thinking – duality (D), necessity (N), and repeated reasoning (R). Their research demonstrated that students who had opportunities to reason and explain, as opposed to students who only practiced computation, were able to discover a mathematical rule for themselves. Their study identifies some myths regarding mathematics education, which unfortunately describe what happens in classrooms everywhere on a regular basis. They explain that these commonly held beliefs have been proven to NOT help develop mathematical thinking:

- When sequencing instruction, begin with what is easy
- The best mental model is a simple one, preferably one quite familiar to the students.
- Use mathematical proofs to convince students that a mathematical result is certain.

Williams looked at abstraction in relation to spontaneous learning, which means learning not caused by the teacher. She created the Spontaneous Abstracting Model and stated that "deep understanding can result from spontaneous abstracting." She advises that, "this type of learning could lead to more connected understandings than guided learning." (Williams 2007) Her model embraces exactly what we hope will happen for most children as they work with the Montessori mathematics materials, and she says that the question of spontaneous learning and abstraction should be further invesitagated.

Simon attempted to refine what happens during the "construction" phase defined in the Nested RBC model. Based on students following a carefully sequenced set of maths tasks, they analysed the results and hypothesised some key aspects of the thought process. They noted that part of the constructive process appears to happen when the learner is not engaged with the activity itself, but outside of the "directed" period. An important conclusion was that focus of attention during the student's activity requires more research to explain what's happening inside the learner. (Simon et al 2010)

<u>In summary, the relevant conclusions about abstraction are:</u>

- There is a difference between simple and constructive abstraction; mathematical abstraction is an example of constructive abstraction
- Constructive abstraction is a progressive reorganisation of the mental processes
- There are 3 steps in the abstraction process: construction, recognition, and building-with; the construction "step" is the key to abstraction
- Teaching does not cause learning
- Part of the constructive process appears to happen when the learner is not engaged in the specific activity
- What occurs in a child's concentration during an activity is not fully understood
- When students have opportunities to reason and explain, they are able to discover mathematical rules for themselves
- There are some commonly held beliefs that do not help develop mathematical thinking

<u>Mathematics and Manipulatives</u>

It's important to recognise that the maths materials we use in a Montessori classroom, whether in a 3-6 or 6-12 class, encompass a carefully sequenced set of ideas and activities that relate to and build on the previous work. Our materials represent a complete body, not a pick-and-choose shopping list for interesting ways to expose children to mathematical ideas. However, research on children using manipulatives for mathematical learning stands outside this framework. The materials used in those studies usually exist in isolation, and relate only to the specific concept being examined.

In a 2001 study, Kamii, Lewis and Kirkland investigated the usefulness of a few specific manipulatives in children's construction of mathematical knowledge. They questioned which manipultives, how to use them, and why. The specific manipultives studied were tangrams, counters and playing cards, and they were examined for their value as a "teaching tool" with which children could (but aren't expected to) learn maths. They found that:

- Tangrams can be useful if children are encouraged to solve problems, but can be reduced in value when adults "help" – for example, adults do the thinking for a student through suggestion
- Counters can be helpful, but interfere with children's thinking if used in an overly prescriptive way; children were often encouraged to use the counters longer than necessary
- Cards could facilitate children's thinking in the specific activity studied

They summarised that the value of these manipluatives depended on how children used them to solve problems. That is, the children had to be thinking for themselves, not assisted by an adult, in order to solve a problem. When the materials were used to help solve that problem, or to stimulate ways of thinking about the problem, they were useful. But when the children were expected to use the materials in lieu of thinking, they were not helpful.

The Montessori maths materials are more sophisticated in their design and presentation than tangrams or counters, but that doesn't preclude an adult becoming prescriptive – such as insisting that a material continue to be used in the belief that more practice is needed with a certain concept. This is especially common when a child comes into the class with some information missing – such as the concept of division. Where do we start? The teaching tendency is often to step back to the simplest material and present the concept, expect repetition, and then gradually move forward through the sequence, regardless of the child's age and the material's intended use. The thinking is that this might also help children internalise the concept of the decimal system (Golden Beads) and the colours used for the hierarchies (Stamp Game, Racks and Tubes). But the above results indicate using the material would only be helpful if it's alligned with the child's conceptual development, and that trying to reinforce those other ideas through repetition would slow down learning division, the goal.

McNeil et al (2009) also studied whether using concrete objects help or hurt a learner in constructing mathematical knowledge. In this study, the topic was using objects to assist children in completing word problems about money. Several different version of objects were used, with some being more realistic ("perceptually rich") than others. The results indicated that the more realistic the object, the less likely the students were to succeed in the given computation. The relationship between successful performance and design of the material is highlighted, and the Pink Tower is used as an example of "bland" design. The authors note that rich design detail becomes redundant once real-world knowledge is activated, and can deter success because, "redundant information is difficult for problem-solvers to ignore." They note that, "the problem-solvers' attention may be drawn toward the redundant surface features of a symbol and away from the abstract ideas being represented by the symbol." Their results endorse the simple and elegant design of our Montessori materials, which isolate the concept being examined.

Another discussion in this same paper is how to "activate real-world knowledge" in a student. It has been shown that "students perform better when working with a real world-activity and/or word problems, than in a purely symbolic context," (DeFranco and Curcio, 1997) so the conclusion is that it's critical to activate a student's real-world knowledge. Unfortunately, learners worldwide have trouble doing this, so manipulative materials have been seen as one way to help activate the thinking process. (Carpenter et al 1980) Specifically with solving word problems, "One promising method involves immersing students in a classroom culture in which problems are treated as modeling activities in which students work together in groups with the teacher to understand the context, generate plausible contextually relevant aproaches, and discuss the merits of different approaches." (Gravemeijer, 2002)

Uttal, Scudder and DeLoache (1997) examined some of the claims about children's use of manipulative materials, such as:

- they enable learners to connect abstract concepts to real objects
- educators around the world have found that children learn (maths) better when using these materials
- use of materials can provide a "cure" for maths anxiety

and found that research does not support any of these claims – again, in relation to the "pick and choose" varieties used in mainstream education. Based on this information, they looked in detail at how manipulative materials are used with children, and concluded that "successful use of manipulatives depends on treating them as symbols rather than as substitutes for symbols."

Their research began from a task in which both 2.5 and 3-year-olds were oriented to a model (concrete material), then asked to complete a real-world activity based on what they had experienced with the model. The results showed that the younger group often didn't grasp the relationship between the model and the real activity; that with poor orientation to the model, the older group performed only as well as the younger group; and that by delaying time between orientation to the model and a real activity, the older children perfomed dramatically less well. From this, the authors concluded that mental maturation is one factor in using materials successfully, and that by carefully supporting the learner, a material can be better understood as a symbol. We might extrapolate then, in our own work, that the introduction phase of work is crucial, and that children need a chance to apply the knowledge of the “model” immediately to a meaningful task. The authors also predicted that making the model more “salient” may decrease the learner’s ability to see the model as a symbol, and allowing children to “play” with a model beforehand (use it for other than its intended purpose) was what decreased their ability to succeed in seeing it as a symbol. Again, extrapolating to the Montessori classroom, this supports the design of our materials and tells us that a material should be introduced with clarity and not used until some introduction to its purpose has been given.

In further discussion, the authors zeroed in on some common problems children encounter with manipulatives. These include the difficulty of seeing the model as related to a concept, and the fact that manipulatives are often seen as a “tool for self-guided instruction.” They cite studies showing successful use of manipulatives when the teacher’s role is to explicitly connect the child with the material, when the material is used as a bridge toward abstraction. It’s never assumed the children will make the connection independently. This disagrees with our expectation that, through repetition and correct manipulation, many children will come to abstraction on their own. The authors also indicate that material with a “systematic internal structure” related to the concepts (such as our carefully sequenced materials) may help children focus on the object-symbol relationship. In summary, the study concludes that when teachers are aware of the potential difficulties children have in grasping materials as symbols, they may be able to forsee and eliminate those roadblocks to understanding.

<u>Summary of the conclusions about concrete materials:</u>

- Successful use of manipulatives depends on treating them as symbols, not as substitutes for symbols.
- By carefully supporting the learner, a material can be better understood as a symbol.
- Children must be thinking for themselves to solve a problem with a material. Materials are useful when they are used to help solve a problem or to stimulate ways of thinking about the problem.
- Rich design detail can deter success by presenting redundant information and distracting from a material’s symbolic meaning.
- Mental maturation is a factor in using materials successfully.
- The teacher’s role is to explicitly connect the child with the material, and the material is used as a bridge toward abstraction; children are not expected to make the connection independently.

<u>Research-Based Teaching Models for Mathematics Education</u>

The National Council for Teachers of Mathematics (NCTM) is a U.S. organisation whose mission is to support teachers to ensure high quality mathematics teaching and learning. The NCTM creates research-based standards, and many studies investigate their implementation.

"The NCTM standards stipulate that students need opportunities to communicate math ideas and solve problems with others, that they should engage in mathematical acitivities with confidence and enthusiasm, and that teachers should use assessment strategies that focus on understanding rather than on right answers." (Stipek et al 2001) This approach is known as constructivist or inquiry-oriented and is very different from the typical focus of teaching maths procedures from a textbook. It's not an easy shift for many teachers, who tend to view mathematics as a static body of knowledge involving rules and procedures that yield a correct answer.

The constructivist theory of teaching and learning describes Montessori educational philosophy. This includes the ideas that:

- Learning involves the active constuction of knowledge through personal experience (Ernest 1994)
- Learning doesn't occur in isolation and is not fixed; it's expressed through language focussing on explanation (Ernest 1994)
- Learning is enhanced through collaborating with others and moving from assisted to independent work (Vygotsky 1978)
- Assessment is consistent with learning principles (Yackel et al 1992)

Constructivist teaching requires a high level of confidence based on good content knowledge, and one study found that "teachers with more traditional beliefs about mathematics and learning had lower-self confidence and enjoyed mathematics less than teachers who held more inquiry-oriented views." (Stipek et al, 2001)

One though-provoking case study (Van Zoest and Enyart 1998) focuses on the 1991 NCTM standard for "dynamic classroom discourse". The authors state that genuine mathematical classroom conversations are rare. What happens instead is often the adult initiates with a question, the student/s reply, and then the adult evaluates the answer. The purpose of this 3-year study was to assist teachers in developing meaningful mathematical conversations by changing their classroom habits, and the results detail 1 participant's progress. Research and evaluation methods included videotaping and analysing classroom discourse, reflecting on and implementing the discourse goals, videotaping again, and summarising progress.

In our Montessori 6 - 12 classrooms, we strive to create an environment where mathematical conversations and discussions occur, but even during our presentations with small groups of students, we could easily replicate the mistakes this teacher found herself making:

- The adult spoke a lot
- The students spoke less than the adult
- The adult interpreted and restated the students' comments

- The adult "cued" the students – gave away the correct answer by verifying a student's response

During the training course, we advise that children should be asked to check their own work upon completion. In the course of certain processes with the children, we ask them, "Does this make sense?" We also affirm that the classroom should not be silent, but that children should be able to argue points with one another – but how do we get this to happen? Most children will come to the Montessori guide with their completed work, looking for verification. Most children will talk in class, but have to be guided about which conversations are suitable for the classroom. How do we create an environment that explicitly pushes children to exercise the Reasoning Mind with mathematical ideas, and to communicate those ideas with others? What steps or practices could the adult use to create this atmosphere?

A common practice that seems unlikely to help is assigning regular maths work that is turned in for review – either with or without being checked first. In either scenario, the model presumes an adult is the authority and the student should work alone to find a correct answer. Even if discussion about finding the answer occurs between student and adult, this doesn't encourage children to question one another or think and work as a team. So what about assigning work by groups then, another common practice? This would give the chance to talk about solutions together – but how would it promote and support talking through a process and valuing other ways of approaching something?

Based on her original videotape, the teacher in this study set some goals. These included listening carefully to students; asking them to clarify and justify their ideas orally and in writing; deciding when to clarify or model and when to let students struggle with difficulty; and posing carefully worded questions that challenge thinking. All of these appear to be appropriate techniques to assist in developing a Montessori classroom culture of mathematical conversation.

When the teacher began listening carefully to her students, she found they often knew what they were talking about even when she didn't understand them. So instead of assuming they didn't know something and explaining it further, she began asking them to explain what they meant. As a result, she developed a better sense of what a student really wanted to know when they asked her a question. This could also be useful to Montessori classrooms. We value children's input at all levels, but listening is not a skill explictly developed through the training course. It's not hard to imagine a Montessori guide thinking that mathematical information is contained in the materials, and, if something's not clear, we should simply explain what we already understand about the materials to help children "get it".

In asking students to clarify their thinking with words, this teacher found children were often uncomfortable, and it took perseverance to help them realise there are different ways of thinking about a solution. It would have been easy for her to alleviate the discomfort by "prompting" with questions, rather than letting children talk through a process. But her goal resulted in more time being spent discussing ways to think about a problem. Over time, children moved from wanting to talk more about their own ideas to listening to each other and being comfortable to discuss their own approach to finding a solution.

Again, I can see this as a valuable goal to work toward in any Montessori classroom. It's easy, given the design of our materials, to develop a classrom culture of 1-2 children working out various problems on any given material, but this isn't really the same as a lively discussion around mathematical ideas. While the idea of asking children to write about mathematical ideas is contained in our albums, it isn't specifically highlighted as an important step in helping develop mathematical thinking. My own experience is limited to discussing "the rule" with children and then asking them to formulate it in their own words. Based on this new information, I plan to experiment and see how children respond.

The goal of deciding when to provide more information or clarification, when to model or lead, and when to let a student struggle, is related to the teacher's understanding of the materials and trust in the abstraction process. In our Montessori classes, it also relates to the question, Which keys does a given child need in order to reach abstraction? The guide has to be comfortable enough with the maths content to trust skipping a few steps in a sequence or jumping around among areas. This could require time, or just a strong mathematical background.

The teacher in this study stopped telling children whether they were right or not when they found an answer, which made them uncomfortable. Instead, she learned to answer their questions with more questions. She found it helpful to an incorrect answer go. This allowed time for the child to think further, rather than feel deeply committed to something s/he has just spent time on. It also gave the teacher time to think about which further questions would help point the child in the right direction. As the students began to develop more conversation among themselves, they were also able to assist one another in this process, which helped them develop ownership of the problem-solving process.

Finally, the teacher worked to improve her ability to ask questions that generated discussion. She found that, "A few carefully worded questions or problems that can lead to good discussion can be more mathematically productive and reinforcing than pages of repetitive problems done in isolation." If our Montessori goal is mathematical inquiry and investigation, rather than repetitive manipulation, then stressing the importance of posing effective questions also seems important. Effective questions might be word problems based on current classroom activities, but they wouldn't be contained in a set of cards or worksheets that children are required to work through. Effective questions could also be generated by the children in a classroom culture where they are encouraged and supported to write their own maths problems.

For mainstream educators, the constructivist approach specifies beliefs but not practices; Montessori practice upholds these beliefs. A 3-stage study (Smith 2000) aimed to bridge the gap between research and practice. Stage 1 highlighted some interesting ideas around assessment to promote reasoning and communication, and ways to make mathematical thinking more visible. These included:

- observation grids
- revisiting a series of work samples for discussion and self-correction
- the use of concept maps in maths work

In Stage 2, the focus was the quality of teacher questioning to foster a meaningful classroom discussion, similar to the previous discussion. Stage 3 again focused on communcation of concepts in various ways, and returned to assessment. Results from the 3 stages were synthesised to form a framework. The communication aspects of their framework encompassed these steps:

- Guided thinking – by adult
- Verbalising thinking – by students
- Clarifying thinking – students, assisted by peers or adult (reflecting on own progress and understanding)
- Writing thinking – concept maps, explanations, labelled diagrams, presentations

All of these communication techniques and guidelines from Stages 1 and 3 could be fleshed out to provide ideas for helping generate meaningful mathematical conversations in a Montessori classroom.

Another NCTM standard, helping children make connections between classroom content and real-life situations, was the impetus for a project focusing on writing strategies within mathematics learning (Haltiwanger and Simpson 2013). Four strategies were trialed and reported anecdotally, with the result that students developed greater ability to communicate mathematical ideas. One of the strategies was to ask children to create mathematical metaphors, similar to what we might suggest as a follow-up activity: asking children to find and list examples of a concept in the environment – such as lines, or angles.

Having good conceptual understanding of mathematics is described as “fluency” within an area. This is shown by being able to select correct procedures, make predictions about the structure of a solution and, “construct new understandings and problem-solving strategies.” (Richland, Stigler and Holyoak 2012) A study of teaching conceptual structure effectively emphasises the importance of reasoning as part of this process. The authors define reasoning as “any circumstance in which logical conclusions are drawn,” and they state that, “there is growing consensus in both the psychological and educational research literatures that teaching students effectively requires teaching them to reason with mathematics.” The authors also pinpoint the need for students to experience “conceptual struggle”.

They cite a meta-analysis (Hiebert and Grouws 2007) of studies of teaching that examined results from high- and low-achieving countries. The critical difference appeared to be the opportunity for students to engage in activities that provided for making connections through struggle, rather than simply focusing on a sequence of procedures to be remembered and followed. This same meta-analysis indicates that teachers' content understanding is an important factor in offering students an opportunity to make connections. While taking variable expertise into account, students from countries where teachers represented mathematics content as a linear organisation were less successful than students from countries where teachers represented the content as an interconnecting web. (Ma 1999)

Research also indicates that children in mainstream education have positive attitudes towards maths at the start of their school years, but that decreases over time, more strongly in girls than boys. (Helmke 1992) Thus, there was a desire to focus on designing a learning environment that prevents this from happening. Our Prepared Environment and the materials are already "designed", so I was curious to see how it compares to the principles named in the study (De Corte 1995). They are:

- The environment should support cumulative, constructive activity and enhance active learning, even for passive learners.
- The environment should encourage students to self-regulate their learning processes.
- The environment should contain activities and artefacts with cultural relevance and offer opportunities for meaningful interaction around them. Tasks should be related to real problems to which children will need to apply themselves in the future.

If the Montessori materials are being used correctly, then the principle for cumulative, constructive activity is a match. The principle of self-regulation would depend on how the class is run. For example, if children simply see demonstrations and complete assigned maths practice, this is not true self-regulation. But if there is two-way conversation about progress, goal-setting, and awareness of required standards, then this principle would be another match. For the third principle, if the guide has created an environment where the materials are seen to have relevance, the material should have "cultural meaning" within the context of the class. This might require explicit conversations about relating a specific maths activity (material) to life through word problems.

Finally, 4-year longitudinal study looked at how teachers can learn to use children's thinking in maths instruction (Fennema et al 1996). Teachers' knowledge "is a major determinant of mathematics instruction and learning" (Fennema and Franke 1992), but "attempts to enable teachers to change their instruction have often been unsuccessful." (Richardson 1990). This study postulated that, as teachers developed their understanding of how children think mathematically, their beliefs and instruction would reflect the expanded information.

Participants in this study began with informal knowledge of childrens' mathematical thinking. Over the 4-year period they were given support via workshops and classroom visits, both focusing on expanding their understanding of children's mathematical thinking and using the new information to make "instructional decisions." The common themes, which reflect some of the previous studies, were:

- Recognising that children solve problems in a variety of ways
- Offering many opportunities to communicate about their problem-solving
- Adults eliciting the child's thinking process

Workshop facilitators did not offer ways to incorporate the new information about children's thinking, they left it to teachers to experiment. Teachers were grouped by levels (1–4a/b) indicating how much their classsroom practices were guided by children's thinking, and at the end of the 4-year period, 18 of the 21 participants had changed their practices to reflect the new information.

Here is a description of a Level 1 teacher – "Teachers perceived their role as that of a demonstrator of procedures and organiser of instructional environments who enabled children to practice what had been demonstrated. Level 1 teachers usually practiced direct instruction by demonstrating the steps in a procedure as clearly as they could and then having children practice repeating the steps." (Fennema et al 1996) When I ask myself, could this scenario happen in a Montessori classroom, the answer is yes.

Here are some hallmarks of the Level 4 teachers – "instructional decisions … were based on knowledge of the kinds of problems their students could solve, strategies students were apt to use, and the kind of communication the students were capable of." And, "one distinguishing characteristic of these teachers' instruction was their ability to conceptualise their instruction almost continually in terms of the thinking of their students." Lastly, "relationships between mathematics and other subjects in the school were contunually stressed."

This description could also be a Montessori teacher, so what's the difference? How are some of us at Level 1 and others at Level 4? This tells me that making sure a Montessori guide understands how to use a material is as important as making sure s/he also sees the role as more than that. We need to stress one's own thinking about the role will make a difference to the mathematical climate, the children's interest and their learning. But since the teaching vision also requires skills, which take time to develop, the simplest solution would be to locate a good reference – book, workshop or webinar – and recommend that information to trainees for future use when we talk about the need to be lifelong learners.

The discussion of positive results in this study emphasises the increased number of students spending most of their time engaged in "true problem solving." It also emphasises the teacher's changed perception of the role – to provide an environment where children's knowledge could develop while engaging in problem-solving and communicating about what they've done. The teachers saw themselves as having an active role in this process by providing suitable problems and by questioning children as they worked through a process. At the end of the study, both concept and problem-solving performance had improved, which the authors believe was related to these changes. Based on this, it appears that stressing the importance of word problems, having a variety of samples in the training environment, encouraging trainees to develop their skills in this area, and providing a few references for them could all be helpful tools.

<u>Discussion: Application of Research Findings to a Montessori Training Course</u>

At the end of this project, I was hoping a clear outline would present itself - of what does and does not help children develop mathematical abstraction, and, therefore, a clear set of procedures to follow. The findings can definitely be grouped within a few general areas, some of which are things I feel we already do, and could emphasise more in the training. I look back on myself as a trainee – the maths work was hard for me, and I lacked confidence. So would any of these ideas have helped me finish with better skills or understanding? If I believe they would have, and can be incorporated into training without changing the delivery of Cosmic Education as a whole, then they could benefit some trainees. Six broad areas are indicated in understanding how mathematical abstraction develops. These are mathematical communication, the importance of struggle, the role of problem-solving, a focus on the teacher's knowledge base, students regulating their own learning, and the purpose of concrete materials.

Mathematical communication is an area that relates to both children and the adult. The more children work in an atmosphere of acceptance and open mathematical communication, the more they will take risks and learn from one another. This has to be modeled and encouraged by the adult, so the adult needs to develop listening and questioning skills. It's important the adult develop an ability to hold the self back, just as Montessori adults do in so many other ways. Rather than verifying an answer, the adult must move to asking questions about how it was found, and listening closely. When the adult doesn't understand, s/he must assume the child knows, and ask more questions for clarification. This process will help the adult understand the child's thinking better, and will help the child regulate his own thinking – he will know for himself when something doesn't make sense, rather than waiting for an adult to tell him. As children reason and explain, they will gradually be "drawing out the rule" for themselves as we discuss in training. It's important that the adult doesn't guide the thinking by prompting an answer. As the adult models listening, children will begin to listen to one another more.

It's just as important for children to have the opportunity to communicate in writing about maths. At the start, this will help children who are less outspoken find their own voice. In addition to listing methaphorical examples of mathematical ideas, labelling diagrams and writing "the rule" in their own words, children might make concept maps of maths ideas they've explored, showing the relationships. Any oral discussions about a process could also be written and then presented to others.

The constructivist idea that children need to "struggle" in order to reach abstraction is a familiar concept for Montessori educators; struggle is what we mean when we talk about self-construction through activity. Can we apply this more precisely to the mathematical thinking process? Making a distinction between computation and struggle would be one possible approach. When we talk about setting challenges for Second Plane children, we might need to focus on what constitutes a challenge. Keeping in mind those common ideas that don't help children develop their mathematical thinking – begin with what's easy and keep the model simple at the start; give proof that an answer will always be a certain way when you follow a given sequence – maybe the issue is, once a concept has been introduced, to recognise that struggle or challenge can't just mean more or larger examples of exactly what has been done already. Since word problems are something we also focus on, and are another way of struggling, maybe they would be what provides repetition with variety, so enough struggle happens in order to assist abstraction.

Problem-solving, as opposed to computation, comes up repeatedly as being important. Problem-solving is identified with a creative thinking process, whereas computation is identified with repeating a procedure. Since children need to be thinking for themselves when engaged, it's important the adult understands the importance of providing problems, offering support at the start, and developing the communication skills that encourage children to work together and stop looking to the adult for verification. Relevant problem-solving has been shown to be the most effective, so it would be important to emphasise developing this material along the lines of incomplete nomenclature. In this way, problems can be a starting point for children each time, and the classroom culture can incorporate an expectation that children continue to create and solve their own problems in any given area.

We are fortunate that AMI training is so thorough and offers the teacher-to-be many avenues for assimilating new information through our face-to-face training. By listening to the theory, watching the demonstrations, participating in practical sessions, creating the albums, and developing observation skills, learners with various strengths have a chance to "struggle" with the information and then demonstrate their understanding. Hopefully, trainees begin a course with an open mind, ready to accept new ideas about children, learning and teaching.

While the research shows teachers' beliefs tend not to change during their training programmes, does this apply to our training? There are many opportunities for reflection about children in our course, but should trainees also learn to reflect about themselves as mathematical learners? Developing this ability during training might translate into greater openness that helps prevent a teacher from substituting his or her own mathematical thinking for the child's. It might also help the teacher better understand the subtle difference between using the maths materials as a means of development rather than a teaching tool.

Since content knowledge is a strong indicator of teacher confidence level, some assimilative writing and mapping activities as part of the preparation for written and oral examinations might help a trainee move from seeing the maths content as linear to seeing it as a web. Deeper content understanding is associated with greater openness and this could lead back to considering how children think at each step, so that maths work becomes a more dynamic classroom interaction.

Compared to any educational method, I believe Montessori classrooms usually offer children more opportunity to regulate their own own learning as a whole. This may or may not be the case specifically with maths, but in cases where it's not, I feel that implementing some of these other research ideas would naturally tend toward creating more self-regulation within maths. Assessment in the mainstream sense is not part of our practice, but the simple ideas of observation grids and revisiting a series of work samples with a student might apply to the individual conferences, especially at the 9-12 level.

Montessori education is recognised by the concrete materials we use, and it's easy for somone without training to think the materials are an end in themselves. The harder task is to make sure a trainee understands how to use the maths materials correctly without attaching to them as artefacts. Attachment to the materials can demand their use, so that children are required to use a material when their mind no longer requires the support it provides. The introduction to the maths album would be the best place to stress this – that children need to see the material as a symbolic representation, and our job is to watch carefully, so we can tell when each child is ready to work with the symbol on its own.

In closing, I have created some questions for myself to keep in mind as I deliver the maths lectures during my first course. I hope they will help me as I continue to grapple with this topic.

<u>Questions about Montessori training and mathematical abstraction:</u>

- Does the training format immerse one "enough" to undergo change of vision regarding children and maths learning?
- At the end of the course, is it possible to have developed the required confidence with the maths materials to have a deep enough understanding to view the maths content as an interconnecting web, rather than a linear sequence? Can this be monitored in some way during the training? Can student reflection be used to assist this?
- Beyond use of the materials, how can training help students face challenges once they go into the classroom – especially from parents who expect an

emphasis on performance rather than mathematical thinking? What practices could assist? What aspects could be better emphasised?

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